

Superfluid flow above the critical velocity

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Superfluidity and superconductivity have been studied widely since the last century in many different contexts ranging from nuclear matter to atomic quantum gases. The rigidity of these systems with respect to external perturbations results in frictionless motion for superfluids and resistance-free electric current in superconductors. This peculiar behaviour is lost when external perturbations overcome a critical threshold, i.e. above a critical magnetic field or a critical current for superconductors. In superfluids, such as liquid helium or ultracold gases, the corresponding quantities are critical rotation rate and critical velocity, respectively. Enhancing the critical values is of great fundamental and practical value. Here we demonstrate that superfluidity can be achieved for flow above the critical velocity through quantum interference induced resonances. This has far reaching consequences for the fundamental understanding of superfluidity and superconductivity and opens up new application possibilities in quantum metrology, e.g. in rotation sensing.

The breakdown of superfluidity and superconductivity above the critical velocity and the critical current [1, 2], respectively, is caused by the production and growth of excitations. In a superfluid, when its flow velocity, or equivalently the velocity of a defect dragged through the fluid, exceeds the critical velocity v_c the creation of excitations becomes energetically favourable. This destroys the frictionless motion, as shown in experiments with superfluid helium [3], and ultracold bosons [4] and fermions [5]. The critical velocity is defined as the maximum velocity below which there is no (or more precisely a bounded) production of excitations. While (essentially) no excitations are present for subcritical velocities, a fast onset of growing excitations occurs for supercritical velocities that gradually decreases for further increased velocity until the kinetic energy of the fluid becomes so high that it dominates all other energy scales and any defects become unimportant again (Fig. 1).

In general, the rate of excitation growth depends on the microscopic details of the superfluid and on how it is coupled to the environment. The details of this coupling determine the dissipation mechanism causing the creation of excitations and ultimately the critical velocity v_c . A simple way to estimate v_c and give a qualitative explanation of the breakdown of superfluidity is provided by the Landau criterion [6].

The Landau criterion is a cornerstone of our understanding of the dynamical behaviour of superfluids, stating that a superfluid flow is sustained against external perturbations or defects up to a critical value of the velocity [6]. Its elegance, power and usefulness rely both on simplicity and generality: there is no need to know the specific nature of the perturbation or the characteristics of the defects, no need to know the microscopic details of the superfluid, and no need to compute the ex-

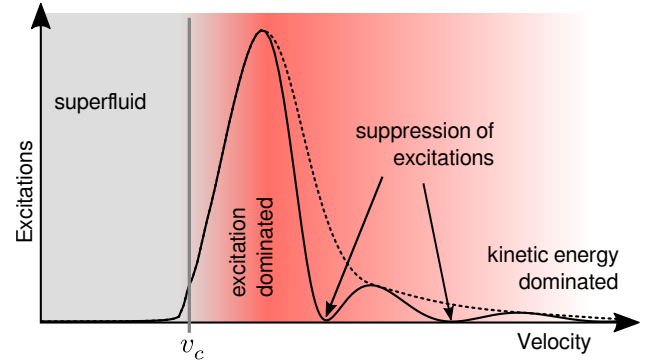


FIG. 1. Excitations in a superfluid as a function of velocity of a defect dragged through the fluid at a given time. For velocities below a critical velocity ($v < v_c$), the production of excitations is suppressed and superflow persists: at longer times the amount of excitations remains very small. At higher velocities ($v > v_c$) a sharp onset of excitations (growing with time) destroys the superfluid properties and only at very high velocities ($v \gg v_c$) the kinetic energy becomes so high that the defect hardly affects the flow. This standard picture (dashed line) has to be adjusted when resonant quantum interference reinstates superfluidity with fully suppressed excitation growth at a series of discrete supercritical velocities. The solid line shows an example for a rectangular defect shape (for the same plot at different times see Fig. 2).

citation spectrum of the moving system; only knowledge of the low-energy excitation spectrum $\epsilon(p)$ of the system at rest is required. Briefly, by applying a Galileo transformation to the co-moving frame it can be shown that for $v < v_L$ (where $v_L = \min \frac{\epsilon(p)}{p}$ is the Landau critical velocity [6]) the production of elementary excitations is energetically unfavoured. From the Landau criterion

it follows that for short-range weakly interacting Bose gases, the critical velocity is equal to the sound velocity c , so that here “supercritical” means “supersonic”. Superfluidity in a weakly interacting Bose gas flowing at a velocity greater than the Landau critical velocity was studied in [7]. Although the detailed analysis of different superfluid systems, including helium and ultracold gases, shows that the Landau criterion often quantitatively overestimates the actual critical velocity v_c , especially in the two- and three-dimensional case (see also a recent paper on one dimension [8]), the identification of a critical velocity above which the production of excitations destroys the superfluid motion is a criterion of paramount clarity and relevance.

The Landau criterion is based on a perturbative treatment of “small” defects affecting the superfluid motion. The possibility to explore superfluid motion in a non-perturbative regime of parameters has attracted considerable interest, e.g. in non-perturbative and/or exact studies of the dynamical propagation of a superfluid in presence of “non-small” defects (of tunable shape and intensity) or periodic potentials [9]. The point we address in this paper is the very surprising possibility of stable *supercritical* propagation (i.e. *above* the critical velocity) at a non-perturbative level. In the following we show that it is possible to have a set of velocities *larger* than the critical velocity v_c where superfluid flow exists and the production of excitations is completely suppressed. The key mechanism here is a recurring resonant phenomenon between the nonlinear wave propagation of the superfluid and the defect. The resulting scenario is plotted in Fig 1 as a solid line, in contrast to the very general notion of the absence of superfluidity for $v > v_c$ (dashed line).

Superfluid motion in presence of defects has been extensively investigated in ultracold atoms: superfluidity can be probed by stirring a laser beam [4, 10–12] and the critical velocity has been measured [4]. Experiments on superfluid motion have been performed also with moving optical lattices [13], in toroidal geometries [14, 15], with ultracold fermions near unitarity [5] and in two-dimensional Bose systems [16].

To illustrate the existence of supercritical flow and the associated arising phenomena we choose defects of rectangular shape in the one-dimensional (1d) flow of a superfluid, whose behaviour is governed by the Gross-Pitaevskii equation. We consider a homogeneous system with stationary flow at velocity v in an initially flat potential, in which a rectangular defect is then introduced to study the dynamical response. From a theoretical perspective, a challenging point is to define the transmission and reflection coefficients since the usual definitions used for linear matter waves [17, 18] do not apply anymore; in fact the superposition principle of an incoming and a reflected wave is no longer valid, and bound states in the defect can be present due to the interaction term [19]. For the interacting gas one can quantify the trans-

mitted part of an incident wavepacket [20–26] or characterize the breakdown of superfluidity caused by the drag exerted by a matter wave on an obstacle [27]. In presence of a δ -like potential moving at supersonic velocity the stationary wave patterns have been studied [28]. When the barrier is rectangular, solutions of the time-independent Gross-Pitaevskii equation can be written in terms of Jacobian functions [29, 30]: for a rectangular well the current-phase relation for subsonic motion can be determined [31, 32] and when neglecting the mean-field interaction outside the potential well, it is possible to analytically calculate the transport properties of the system in terms of incoming and outgoing waves and resonances and bound states are obtained in closed form [33]. Understanding matterwave propagation in the presence of tailored defect potentials is important for the study of analogue gravity models and acoustic Hawking radiation in Bose-Einstein condensates [34].

It is well known that in the non-interacting regime the transmission coefficient across a square potential, as found by solving the time-independent *linear* Schrödinger equation, reaches exactly unity for specific values of the momentum of an incident plane wave (Ramsauer-Townsend resonances). In this work we quantify the creation of excitations, given the geometric parameters of the defect, in relation to the flow velocity v and the interaction strength g in the gas. We find that the resonant behaviour has a counterpart in the *nonlinear* regime and superfluid motion across a defect is possible in spite of the Landau criterion predicting growth of excitations. The velocities at which this excitation-free supercritical flow occurs are shifted and continuously connect to the velocities of unit transmission in the linear case as the mean-field interaction parameter is varied. This can be viewed as generalised Ramsauer-Townsend resonances that continuously connect the non-interacting case to the attractively and repulsively interacting regimes.

Given the explicit arguments of the Landau criterion it is important to clarify what is meant by supercritical superfluid flow: *i)* A supercritical solution of the dynamical equations exists for specific values of the superfluid’s momentum and system parameters (i.e., interaction strength and geometric properties of the defect) and the flow of such a solution is stable under small perturbations. *ii)* The perturbations of the propagating flow created by the defect is bounded in time (at least for experimental timescales) and no new excitations are produced when the barrier is completely ramped on. *iii)* The superfluid flow is physically inducible and accessible. We have found that all three conditions are fulfilled for the supercritical flow discussed here.

We have verified that the production of excitations at the supercritical excitation-free points is bounded in time, that the flow is stable under small perturbations and for times larger than the typical experimental timescales of ultracold atom experiments with 1d Bose

gases, and that the obtained findings do not crucially depend on the ramping time of the well/barrier.

The Gross-Pitaevskii equation we consider is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x, t)\psi + g|\psi|^2\psi \quad (1)$$

where $\psi(x, t)$ is the condensate wavefunction, g is the one-dimensional nonlinear coefficient [35], and the potential is chosen to be

$$V(x, t) = \begin{cases} V_0 \tanh\left(\frac{t^2}{\alpha^2}\right) & \text{for } 0 < x < d \\ 0 & \text{otherwise} \end{cases}, \quad (2)$$

where d is the width and V_0 the strength of the defect, which is ramped on at time $t = 0$ with a speed parametrized by α . At time $t = t_{\text{barrier}} \equiv 1.5\alpha$ the value of the defect is $0.98V_0$, so it is practically almost completely turned on. The initial state (when the barrier is absent) is a plane wave with momentum k and velocity $v = \hbar k/m$: $\psi(x, t=0) = \psi_0 e^{ikx}$ and density $n \equiv |\psi_0|^2$. Since for $t = 0$ the defect is absent, $\psi(x, t=0)$ is a solution with momentum k of the time-independent Gross-Pitaevskii equation. The idea of ramping on the defect is to adiabatically lead the system towards possible supercritical superfluid solutions for specific values of the momentum k .

We consider periodic boundary conditions on a domain with finite length $2L \gg d$ (with $x \in [-L, L]$), and we check that for the considered times the effects of raising the barrier do not propagate to the boundaries; thus, the results are independent of the specific choice of boundary conditions. For the numerical solution we used the projected fourth-order Runge-Kutta method in the interaction picture [36], which we confirmed to be very stable for all times within the considered range.

To quantify the production and growth of excitations we found convenient to introduce the time-dependent quantity $\mathfrak{D}(t)$ (which we refer to as the disturbance) defined as

$$\mathfrak{D}(t) = \int_{-L}^0 dx (|\psi(x, t)|^2 - |\psi(x, t=0)|^2)^2, \quad (3)$$

where the integral is calculated over the region where the initial wave propagation is directed towards the defect. $|\psi(x, t=0)|^2$ is the density n . Of course, if the defect is absent $\mathfrak{D} = 0$.

Our results are summarized in Figs. 2-4. Fig. 2 shows that essentially no excitations are produced for velocities below the sound velocity $c = \sqrt{gn/m}$ (here $v_c \approx v_L = c$). As expected, excitations are produced for $v > v_c$, as time progresses $\mathfrak{D}(t)$ increases, and for large velocities $\mathfrak{D}(t)$ becomes smaller. However, there are velocities $v > v_c$, for which the production of excitations is inhibited. As shown in the inset of Fig. 2, close to these points the production of excitations is bounded in time. Panels (a),

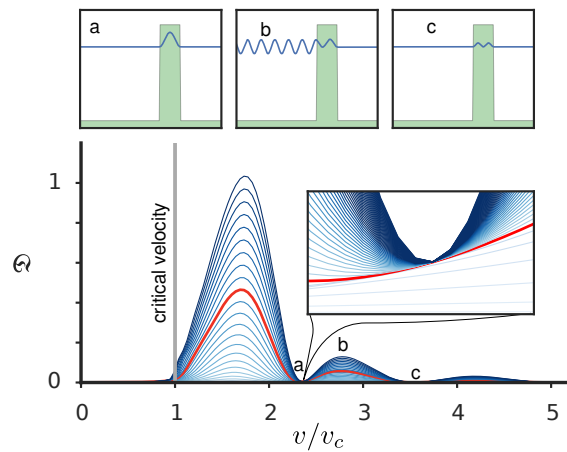


FIG. 2. Disturbance \mathfrak{D} vs v/v_c at various equidistant times (growing from the bottom of the figure), where v_c is equal to the sound velocity c . Darker lines indicate longer times. The line highlighted in red corresponds to $t = t_{\text{barrier}}$. At specific values of v , the disturbance does not grow with time after the barrier has finished rising. The insets illustrate this phenomenon in detail for the first minimum. Panels (a),(b),(c) show the density $n(x) = |\psi(x, t_0)|^2$ (in blue) and the potential $V(x, t_0)$ (in green) at time $t_0 = 2.2t_{\text{barrier}}$ for three different initial values of v corresponding to the minimum and maximum points for \mathfrak{D} indicated as “a”, “b”, “c” in the main figure. The initial condition is a plane wave travelling with momentum $\hbar k = mv$ towards the right. The simulation parameters are $g = 15 \times (\hbar^2/2md)$, $\alpha = 3 \times (2md^2/\hbar)$ and $V_0 = 2 \times (\hbar^2/2md^2)$.

(b), (c) of Fig. 2 show the density at a fixed time larger than t_{barrier} . Away from the minima phonons are emitted (Fig. 2b) while at the minima a stationary breathing state forms inside the defect (Fig. 2a and Fig. 2c)[27]. We checked that these results neither depend on the particular choice of the measure of disturbance considering other quantifiers of the excitations, nor on the periodic boundary conditions or the choice of L . These findings also do not critically depend on the value α . This is true for typical experimental values of α as well as for $\alpha \rightarrow 0$, where small fluctuations of $\mathfrak{D}(t)$ occur for specific choices of quantifiers of the disturbance. We finally observe that our results hold for a wide range of interactions, including large values of $g > 0$, so that the validity spans the regime where the healing length is smaller than the defect width to the regime where it is larger. For small negative values of g the resonances are still present, but when $-g$ becomes large then collapse and instabilities are observed as expected.

As shown in Fig. 3, there is a clear difference between the growth of $\mathfrak{D}(t)$ at the resonant and non-resonant velocities. At resonant velocities, excitations are produced exclusively during the ramping of the defect and the disturbance is afterwards constant. For non-resonant velocities the disturbance grows linearly with time, which leads

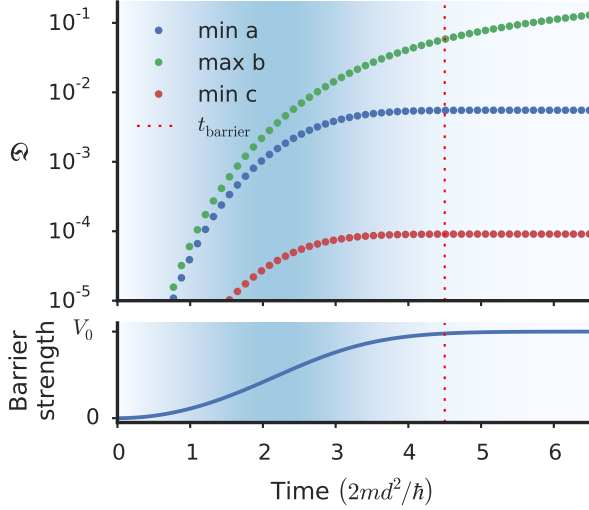


FIG. 3. Disturbance \mathfrak{D} as a function of time for the initial velocities v corresponding to the minima and maximum indicated in Fig. 2 (top) - the bottom plot shows the corresponding time dependence of the barrier strength. The background shading indicates the rate of change of the barrier strength. For times $t > t_{\text{barrier}}$, where the barrier strength has settled, \mathfrak{D} has a linear dependence on time. The creation of excitations can be characterised by the slope of the disturbance $\dot{\mathfrak{D}}$ at large times and it is extremely small at the minima (see central right panel in Fig. 4 for details).

us to quantify the growth of excitations by computing the time derivative of $\mathfrak{D}(t)$ for $t > t_{\text{barrier}}$ (and checking that the obtained value does not depend on the computation interval). The analysis shows that $\dot{\mathfrak{D}}$ is extremely small at the resonant velocities (being suppressed by at least 4 orders of magnitude with respect to non-resonant velocities). $\dot{\mathfrak{D}}$ is also very small for $v < v_c$ in agreement with the Landau criterion. In the inset of Fig. 4 we plot the values $\dot{\mathfrak{D}}$ as a function of the velocity for four interaction strengths. By performing the same analysis we can reconstruct the behaviour of the resonant velocities v_{res} as a function of g : the results are plotted in Fig. 4 for a barrier (positive V_0). With respect to the non-interacting limit $g = 0$ the shift is positive (negative) for repulsive (attractive) interactions $g > 0$ ($g < 0$). Similar results hold for a potential well ($V_0 < 0$), then with a negative (positive) shift for repulsive (attractive) interactions. Our results are plotted in Fig. 4 together with a multiple-scale analytical derivation valid for small g , predicting values of k_{barrier} (defined below) corresponding to a total transmission across the barrier [37]. These momenta are given by $k_{\text{barrier}}d = n\pi + \delta$, where $k_{\text{barrier}} = \sqrt{k^2 - \frac{2mV_0}{\hbar^2}}$ (with velocity $v = \hbar k/m$) and

$$\delta = \frac{3a(kd + n\pi)}{8} \quad (4)$$

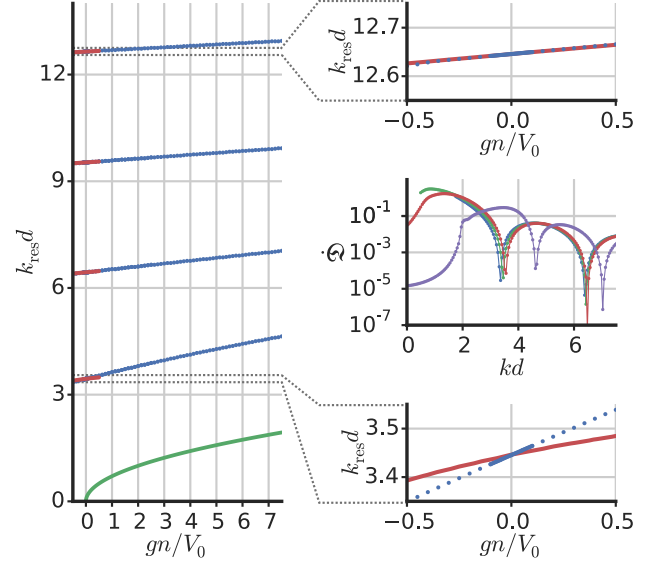


FIG. 4. Resonant momenta (with $mv_{\text{res}} = \hbar k_{\text{res}}$). Centre-right: disturbance growth rate $\dot{\mathfrak{D}}$ as a function of kd for four different values of interaction: $gn/V_0 = -0.5, 0, 0.5, 7.5$ (blue, green, red, purple lines, respectively). Left: Resonant momenta $k_{\text{res}}d$ as a function of gn/V_0 for the first four excitation-free points. The solid red line is the perturbative prediction (Eq. 4), the blue points indicate numerically calculated values and the green line shows the sound velocity $c = \sqrt{gn/m}$. Bottom-right and top-right: detailed view of the first and fourth resonances, respectively.

with $a = 2mg/v^2$. Our result differs by the factor $(-1)^n$ from Eq. 24 of [37]. The analytical predictions for small g match the numerical data well for the higher excitation-free points, but less so for the lower resonant velocities. The ratios of the slopes of v_{res} as a function of g between the analytical and the numerical results are 0.47, 0.83, 0.92 and 0.95 for the first four excitation-free points in increasing velocity order.

An experimental setup to test these results can be implemented with ultracold Bose gases trapped on an atom chip. A 1d quasi condensate can fill a potential tube of several hundred micron length [38] with a very small potential variation along the tube and correspondingly homogeneous density. The gas can be set in motion at a controlled velocity by removing the residual longitudinal confinement and applying a short pulse of a magnetic gradient in the same direction. Velocities on the order of and exceeding a typical sound velocity of $c \sim 1$ mm/s can be achieved straightforwardly. By applying currents to microwires on the chip, a magnetic defect can be produced and controlled. Its geometric shape can be tailored with a resolution given by the atom-surface distance z_0 . Here it is critical that z_0 is on the order of single microns in order to distinguish the individual excitationless resonances from the intermittent regimes of fast excitation growth. The excitation behaviour can be probed by vary-

ing the velocity v of the gas's motion or by varying the final amplitude of the defect V_0 at fixed v . The difference in V_0 for the first two excitation-free points is expected to be $\Delta V_0 \approx \frac{3\hbar^2}{2m} \frac{1}{d^2} = h \times 6.9 \text{ kHz} \times \mu\text{m}^2 \frac{1}{d^2}$ for the example of ^{87}Rb , so that $d \gtrsim z_0$ needs to be sufficiently small to maintain $\Delta V_0 \approx \mu \approx h \times 1 \text{ kHz}$, where μ is the chemical potential of the repulsively interacting gas. Appropriate rise times of the defect are on the order of milliseconds ($t_{\text{barrier}} \gtrsim 1\text{ms}$) for $z_0 \gtrsim 1\mu\text{m}$.

In conclusion, we have studied the propagation of matter waves across defects of rectangular shape starting from a stationary flow solution in the defect-free case with velocity v and ramping on the defect: for velocities smaller than a critical velocity v_c there is no production of excitations (with v_c very well approximated by the Landau critical velocity v_L in our one-dimensional case). For a set of supercritical velocities $v > v_c$ the growth of excitations is fully suppressed, contrary to the generic expectation as a consequence of the Landau criterion. For these velocities, we find the production of excitations to be bounded in time and to stop entirely when the defect is completely turned on. The flow is stable for times far in excess of any typical experimental timescale in the domain of ultracold atoms and elongated Bose condensates in particular. The obtained findings do not crucially depend on the ramping time of the defect. Such excitation-free supercritical velocities are present both for wells and barriers, and for repulsive and (small) attractive interactions. We observe that even though in the nonlinear case bound states and bifurcation effects are expected, our protocol allows us to access the excitation-free points in a clean way, not depending on the ramping time.

The obtained excitation-free supercritical velocities are the nonlinear counterpart of the velocities having unit transmission in the linear Schrödinger case (Ramsauer-Townsend resonances) and are due to the resonance between the length scale associated with the matter wave momentum ($\sim 2\pi/k$) and the length scale of the defect. The shift from the resonance is positive (negative) for repulsive (attractive) interaction in the case of barrier defects, and vice versa for well defects. Beyond this surprising proof of the presence of such excitation-free supercritical velocities for the paradigmatic case of rectangular defects, we expect that they exist in general whenever a defect can be characterised by a well defined length scale, e.g. for trapezoidal defects or two delta-peaked potentials. The steeper the defect is at its edges, the more robust will the inhibition of excitations be in the vicinity of a set of supercritical velocities.

Motivated by our work it will be interesting to study general criteria of existence and stability of supercritical solutions of the nonlinear Schrödinger equation in presence of defects with a well defined length scale. On the other hand, intriguing possibilities arise from utilising supercritical points in measurement devices based on superfluids and superconductors. Tuning a barrier to

a supercritical flow resonance would facilitate precise determination of unknown external parameters affecting the flow velocity such as rotation and performing selective measurements at supercritical velocities.

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